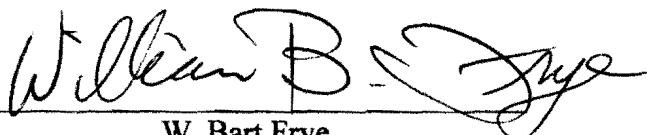


Retirement Mathematics

An Honors Thesis (HONRS 499)

by

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Abstract

This paper discusses retirement and the mathematics pertaining to retirement. There are many approaches to discussing retirement including discussion of vehicles, strategies, implementation, and failure. While each of these topics is voluminous by itself, this paper attempts to accumulate relevant knowledge from each topic and present it in a manageable form. The result is a mathematical and actuarial consideration of retirement. Further, this paper brings high-level topics to a level acceptable to general readers in order to highlight the importance and impact each topic has on individual retirees.

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Retirement. This foreboding noun pervades the homes and offices of the global workforce, young and old. “When should I retire?” “How much money should I save?” “What kind of investments should I have?” “How much money will I spend?” These and other questions like them are daunting to any individual concerned with his or her future financial welfare.

While these questions are highly subjective in nature, there are many objective considerations often overlooked in the course of making decisions related to retirement. As with all aspects of life, choices for individuals are made by individuals and will ultimately be determined by their needs and preferences. However, a mathematical analysis of some features of retirement will hopefully bring a higher awareness of the long-term impact such choices may have on retirement.

The goal of this paper is not necessarily to present new formulae or theoretical conceptions of retirement mathematics. Rather, the intent is to present mathematical concepts related to finance and retirement and to draw meaningful considerations from these concepts. Specifically, this paper is designed as a resource for individuals who have a basic knowledge of financial mathematics. Hopefully this paper will provide some mathematical insight into pension and retirement mathematics for the actuarial student as well as provide an informed overview to other readers. By presenting basic characteristics, ideas, and applications of retirement mathematics, readers may be better informed on the mechanics of retirement. The paper concludes with a brief discussion of investment strategy as well as a consideration of computational techniques.

Retirement Instruments

Surely retirement instruments have come under higher scrutiny in recent years, due in part to incidents publicized in national media. Pensions and Social Security benefits are commonly important components in employee retirement planning. With full retirement age at 65 – gradually transitioning to age 67 for persons born after 1937 (Full Retirement) – and life expectancy at birth of 77.9 – 75.2 and 80.4 for males and females, respectively, based on preliminary estimates of 2004 data (Miniño) – it could be viewed that an average retiree might receive 11 to 13 years of retirement benefits. However, it is possible to receive social security benefits as early as age 62. Another pitfall of the social security scheme is unexpectedly long lifetimes, with documented cases of over 100 years of age (see Appendix A). Thus, the number of years of benefits received widely ranges from 0 to 40 or more years. While advancements in relevant data analysis techniques – e.g. advancements in mortality studies – have been developed and improved upon over time, the actual cost experienced by providers of such plans remains a significant hurdle to be overcome.

In consideration of the unique investment strategies which individuals may implement, this paper focuses primarily on pensions with occasional reference to Social Security and other investment vehicles. Hereafter, an *employer* will denote any firm, sponsor, or entity that provides the retirement plan contributions and an *employee* will denote any employee, participant, or recipient of retirement plan benefits.

Plan Types: A Simplified Overview (DC v. DB)

To simplify considerations, this section will emphasize retirement vehicles primarily in the categories of defined-contribution and defined-benefit. Although defined-contribution plans have become more prevalent than defined benefit plans in recent years (Moore 158) this paper will still consider both methods of pension construction, if nothing else in order to provide a comparative analysis. For a *defined-contribution* (DC) plan, the actuarial present value can be defined as “the accumulation under interest of contributions made by or for the participant, and the benefit is an annuity that can be purchased by such accumulation” (Bowers 356). In contrast, *defined-benefit* (DB) plans are usually defined formulaically in terms of compensation levels near retirement.

While both methods share a common goal by seeking to provide secure income streams for retirees, they differ quite drastically in their design. For instance, a DC plan involves an employer setting aside a regular contribution – defined at the inception of the plan – that will accumulate until a corresponding annuity is purchased. Although the employee largely has control over decisions involved with DC plans, the employer retains the responsibility of providing administrative functions as well as overall investment responsibilities. In contrast, a DB plan is merely formulaic. One such formula – the final n -year salary benefit – calculates DB benefit as a function of the final n years of salary, while another calculates benefit as a career average (Bowers 352).

Another central difference between DC and DB plans is the burden of accumulation. DB plans certainly depend on employee performance. For example, a DB plan based on final m -year average salary or career average benefit is unequivocally dependent upon employee performance. However, while DB plans have some employee

input the resultant formula has been fixed since the inception of the DB plan (barring any amendments or alterations in the plan design throughout employment). While employee performance and term of service are significant in DB plans, it is the employer's duty to accumulate sufficient assets in response to expectations about employee service. In contrast, DC plans are not constrained by an employee's performance; rather, they are constrained by an employee's investment decisions. When constructing DC plans, an employer begins with a target benefit that the employer deems a reasonable benefit according to its particular methods and needs. By computing the actuarial present value of the benefits, the employer arrives at the respective contribution which it will "give" to the employee. Thus, the employee has freedom to invest and manage his plan assets – to an extent – until the annuity is purchased. Thus the employee assumes the risk of adequately accumulating the assets to reach his target benefit. An employee will rarely exactly arrive at his target benefits and – depending on assumptions used by the employer – the actual benefits may vary greatly from the intended initial target.

Specific Retirement Vehicles

There are a variety of retirement vehicles available in the market today. IRA, Roth IRA, 401(k), Roth 401(k), Social Security, and pension plans are just a few examples of modes of investment and saving with retirement in mind. Each retirement vehicle has a different structure and intent, as well as a respective degree of employee control. One approach in considering the array of retirement vehicles available is to provide theoretical examples and case studies to weigh the advantages and disadvantages of each plan. However, choosing a "solution" retirement instrument over any other is a personal option decided

by *one* analysis of *one* individual's scenario. Therefore, focus is shifted away from analysis of vehicles and onto analysis of risks.

The Risks of Retirement

Continuing to focus on pensions, risk analysis begins with a discussion of Asset-Liability Management. Asset-Liability Management (ALM) is an aptly named exercise in modern financial mathematics of balancing assets and liabilities. ALM may be formally defined as “the ongoing process of formulating, implementing, monitoring and revising strategies related to assets and liabilities to achieve an organization's financial objectives” (Luckner 2). Essentially, an institution (such as a bank or an insurance company) makes guarantees in return for a benefit of some kind (money, labor, etc.). If an insurer offers too high a benefit, charges too low a premium, or makes poor investment decisions, it will likely have outstanding obligations which it cannot satisfy. In a retirement system, such insufficiency and insolvency are a detriment to retirees who may rely on their retirement vehicles as a means of income. Therefore, it is pertinent to discuss the risks which are important in maintaining sufficiency.

Any company must make positive net gains in order to exist, let alone flourish. A key to such continued existence – and perhaps success – lies in proper ALM. An important quantifier in the definition of ALM is *ongoing*, implying that ALM is not a one-time calculation. Moreover, proper management occurs over the lifetime of assets and liabilities due mainly to the fluctuation of *risks*. Here a *risk* is defined as a measure of volatility of value, obtained either by retrospective analysis based on experienced data or by prospective analysis based on expectations of the market (Luckner 18). In defining a

risk, it should be immediately noted that risk assessment can be carried out and analyzed via multiple methods, in turn leading to different measurements of different types of risks. A few categories of risks include equity, liquidity, legal, currency, and country risk. When balancing assets and liabilities, it is important to note that different implementation scenarios of ALM require consideration of different types of risks.

The mathematics of ALM are primarily concerned with long-term management of assets and liabilities, as opposed to liquidity management which deals with short-term management (Ho 22). When specifically considering long-term management, three primary risks are generally associated with matching risks and returns: interest rate risk, liquidity risk, and risk of ruin.

Interest rate risk concerns the return (interest) on assets and liabilities. Although interest rate risk is usually associated with asset interest, the interest on liabilities is equally important. As market interest rates fluctuate, an institution should implement strategies for managing both sets of interest rate risks and not just the risk on assets. Liquidity risk considers the risk of being able to sufficiently liquidate funds to meet any term of obligation. Finally, the risk of ruin is concerned with bankruptcy, insolvency, and similar conditions of transactional counterparties. An insurer deals with this risk within investment portfolios and faces similar risks – mortality risk and catastrophe risk of insureds – on their liability side as well.

While a variety of risks can have meaningful impacts on valuation of assets and liabilities and risk reporting functions, not all portfolios will experience all types of risk. Therefore, the following two sections deal with interest rate risk and mortality risk – two risk categories which are especially important to the long-term nature of retirement.

Risks of Interest

Interest rate risk is of prime concern on *both* assets and liabilities. The net interest income (NII) is the resultant gain or loss due to interest rates on assets and liabilities in total (Crouhy 183). ALM provides methods to model and estimate interest-related risk. Decisions taken in response to interest rate risk on assets *and* liabilities help create a correlated environment that hopefully maximizes NII. Thus, NII (the interest-based income) in combination with non-interest related revenue (essentially the fee-based income) create the total revenue that a company may expect during a time period. Non-interest related income are composed of items such as service fees and application fees and are primarily more difficult to model than NII. This difficulty arises due to the unpredictable nature of human tastes, preferences, and behaviors. For example, service fees and application fees associated with creation with and maintenance of policies or mortgages are typically fixed and thus limited by the number of individuals who choose to instigate such transactions. Similarly, late fees and increased rates assessed as penalties are difficult to model because they are based on countless unique factors associated with individual counterparties. In a long-term environment such as retirement, risk associated with such fees and non-interest related financial components are comparatively less interesting than interest-related risk; therefore, development of non-interest related models will not be sought.

A few tools related to interest rate risk include the yield curve, duration, and immunization strategies. The yield curve – a primary source in interest rate evaluation – is the curve for the market of nearly risk-free securities such as government bonds (Ho 27). There exist an assortment of different models for yield curve assembly and

usage; however, their derivation is not a consideration here. It is sufficient to state that each yield curve is a random curve indexed by a fixed time t , giving the values of interest rates of any maturities of length $T - t$. An example of a yield curve obtained from the US Federal Reserve can be found in Appendix B.

The nature of the yield curve has recognizable impacts on retirement portfolios – as well as other investment portfolios generally. Consider a simple portfolio of three bonds: a 2-coupon bond with coupon rate of 4% per period, a 20-coupon bond with coupon rate of 8% per period, and a 60-coupon bond with coupon rate of 12% per period. Assume the initial yield rates for the bonds are equivalent to the respective coupon rates and each bond has a face value of 100. Since the yield rates are equivalent to the coupon rates, the initial portfolio price is simply 300. Now consider a parallel shift in the yield curve of +100 basis points (bps). In other words, each interest rate on the yield curve increases uniformly where 100 bps is equivalent to 1%. Denoting $a_{\overline{n}|i\%}$ as the present value of an annuity-immediate with n periods and interest rate i per period, the new price of the portfolio becomes:

$$P^{(1)} = 100 + 100(.04 - .05)a_{\overline{2}|5\%} = 98.1406$$

$$P^{(2)} = 100 + 100(.08 - .09)a_{\overline{20}|9\%} = 90.8715$$

$$P^{(3)} = 100 + 100(.12 - .13)a_{\overline{60}|13\%} = 92.3127$$

$$P^{(1)} + P^{(2)} + P^{(3)} = 281.3248$$

This shift in the yield curve shows a decrease in purchase price of the portfolio by 6.2%. The price decrease may seem insignificant, but to a large employer funding an extensive pension plan it can mean tens or even hundreds of thousands of dollars.

The above parallel shift in the yield curve simplifies assumptions about yield curve changes. Whereas uniform shifts are uncommon, a flattening of the yield curve is a change more indicative of natural markets. Suppose the yield increases by 150 bps for a 2-coupon bond, decreases by 50 bps for a 20-coupon bond, and decreases by 100 bps for a 60-coupon bond. The new price of the portfolio is

$$P^{(1)} = 100 + 100(.04 - .055)a_{\overline{2}|5.5\%} = 97.2305$$

$$P^{(2)} = 100 + 100(.08 - .075)a_{\overline{20}|7.5\%} = 105.0972$$

$$P^{(3)} = 100 + 100(.12 - .11)a_{\overline{60}|11\%} = 109.9672$$

$$P^{(1)} + P^{(2)} + P^{(3)} = 312.2949$$

A flattening of the yield curve as the above example indicates would likely lead investors – especially large, corporate investors – to change their portfolio profiles.

The nature of the yield curve is important to retirees in that it may shape their retirement outcome. In defined-contribution plans, benefits are dependent on – among other things – the interest rate at which funds are accumulated. While an ideal yield curve has larger interest rates for longer investment periods, a more flat yield curve will have similar interest rates along the curve. Depending on the liquidity of the investment vehicles, a flat yield curve may call for retirement funds – e.g. funds in an employer's pension plan – to be invested in short-term bonds which pay slightly lower interest rates in exchange for the ability for mobile funds that can be reinvested as the yield curve shifts and changes shape.

The Longevity Problem: Mortality Risk

The other main category of risk associated with retirement has become a growing concern – particularly in response to the aging national population of the United States – is mortality risk. Classic insurance models deal with mortality whenever products are non-certain. In fact, the only insurance product that would not consider mortality would have a single premium at time $t = 0$ and a guaranteed future payment(s). Otherwise, payment of premiums and reception of benefits are based on the insured remaining alive. Thus, mortality risk combined with the previously discussed finance-related risks greatly complicates the problem of measuring total risk.

With the impending retirement of the baby boomers, investment security is fast becoming a topic of discussion for employers and fund managers seeking to meet the needs of this older population. Specifically, financial experts seek to have sufficient liquid funding available to pay benefits to retirees. The greatly increased retirement and potential retiree population is the complicating aspect of meeting such sufficiency in comparison with the same problem ten years ago.

A 1995 study of time series data from industrialized nations found that “on average people are working significantly less while living longer” (Ausubel 113). This simple connection is generally not a new or astonishing one. The most recent actuarial study published by the United States Social Security Administration shows the generally increasing trend in life expectancy (see Figure 1). For example, while life expectancy at birth in 1940 was in the 60s, by 1980 it had risen into the 70s and is forecast to rise even higher (Bell).

The implied increased longevity by these and similar studies is both a good and bad thing. A major concern with increased *quantity* of life is *quality* of life. While the average woman may be living close to 80 years of age, the issue of quality becomes increasingly important as she approaches years above and beyond 50 and 60. There are many contributing sources that impact the quality of life: diet, mental well-being, exercise, stress, and spiritual outlook are just a few examples. However, there is one factor at the root of all others that has an enormous affect on the availability of the means to positively increase quality of life – money.

Financial health has the potential to define an individual's lifestyle. This is not to say that more money equates to higher quality; rather, sufficient and adequate finances are needed to fund an individual's lifestyle, be they small or large. Thus, upon retirement individuals naturally expect to continue living up to the standards they previously held as working citizens. Unfortunately this is largely a difficult task. Without a continuing stream of income, a retiree's survival depends on the strength of his or her financial health and security. On top of calculating a continued lifestyle cost, the increasing age of retirees leads to higher healthcare costs, higher medical costs, and at times higher insurance costs. Of course, there are many nuances to consider during the retirement transition, but generally these three are the largest of budgetary concerns.

The consideration of assorted medical costs for retirees reveals a large financial burden to be provided for. An independent survey conducted in 2006 by The Segal Group expects that “[with] average per-participant [health care] costs of \$7,600 (composite single/family before cost sharing), a 12 percent increase would mean plan sponsors could expect a cost increase of over \$900 per participant in 2006, if they maintain current levels

of medical benefits” (Segal 4). Although this projection has decreased from the group’s prior year study (see Appendix C), the *one-year* projected increase is still a significant one. Consider the case of Firm XYZ containing 100 employees whose cost follows the 12 percent trend. With a total health care cost of \$760,000 in 2005, the firm will have an increased total cost of \$851,200 in 2006.

Continuing the example of Firm XYZ, consider the cost of health care in three cases. Case A has a constant inflation rate of 2.5 percent. Case B follows the 12 percent expectation in 2006 and increases only by inflation thereafter. Case C follows the 12 percent expectation in 2006 and increases by inflation at an increasing rate of 3 percent thereafter. A simple analysis shows that in 15 years, the cost of healthcare per capita has the potential to rise between 100,000 and 200,000 per year (see Appendix C). Even considering an ideal scenario that contains no increase to cost – other than inflation – in 2007 and beyond, it can be seen that costs compound rapidly with an inflation rate held constant at the 2006 estimate of 2.5 percent (CIA).

Retirement Strategy: A Solution?

The natural question that arises is: What solutions can be offered to offset the uncertainties associated with retirement? The decrease in mortality rates coupled with the rise in population, the rise in interest rates, and the rise in cost of living will continue to be the causes of this problem in the future; since the causes will not easily be altered, the solution lies in altering the effects – i.e. amending the methods by which retirements are funded. One such method is presented here.

In a 2006 article, Moore and Young seek to develop an optimal investment strategy (Moore). Expanding upon previous work, Moore and Young derive a relationship between optimal asset allocation and ruin probability. They consider a retiree “who does not have sufficient wealth or income to fund her future expenses, [seeking] the asset allocation that minimizes the probability of financial ruin during her lifetime” (Moore 145). In other words, they seek to develop a model to allocate assets such that an individual’s wealth w has the best chance to avoid falling below a lifetime ruin level $w_T \geq 0$ which is able to provide for the individual’s minimum consumption level c . The resultant *optimal investment strategy* is

$$\pi^*(w, t) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(w, t)}{\psi_{ww}(w, t)},$$

where r is the risk-free rate on the (nearly) risk-free investments, and

$$\psi(w, t) = \inf_{\{\pi_s\}} Pr[\tau_l < \tau_d | W_t = w].$$

is the minimum probability that the individual age x outlives her wealth. It follows that the *optimal investment process* is

$$\Pi_t^* = \pi^*(W_t^*, t) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(W_t^*, t)}{\psi_{ww}(W_t^*, t)}.$$

Detailed development of this theory can be found within the Moore and Young article (pp. 147-156). Essentially, this optimal investment process is a function of time t and of the optimally-controlled wealth at time t – i.e. optimal investment is dependent upon a specific time and the wealth accumulated up to that same time. This result is intuitive as an investment portfolio changes over time due to prior investing strategies and current

investing decisions vary as needs change over time. Basic numerical examples are provided as well, giving a general insight into changing investment strategies over time (see Appendix D for one such example).

While this example concerns an individual and her investment strategy, the implications of following a similar – albeit more appropriate and likely more complex – strategy in a group pension plan are apparent.

Computation: A Discussion of Technique

Perhaps an even greater concern for pension mathematics lies in the foundations of such computations. Due to different computational techniques – from actuaries, financial economists, accountants, and others – there is a more broad-sweeping debate on the correct method of specifying value. This debate has revolved generally around valuing assets and liabilities, but the implications a change in valuation may have in pension mathematics alone are enough to consider the argument.

A major consideration in the realm of computational techniques revolves around financial economics compared to actuarial mathematics. In a recent publication of the *North American Actuarial Journal*, actuary Tony Day – Head of Strategy for Queensland Investment Corporation – makes an analysis of traditional actuarial techniques in comparison with and in conjunction with techniques of modern financial economics. From his work, Day surmises that “many standard modes of actuarial thought are, in fact, indefensible when examined with the tools and techniques of financial economics. The call for revision of actuarial training and practices is credible and necessary” (Day 91).

What is the source of such disparity in value? Day presents several problems, all of which revolve around the actuarial consideration of cash-flow. Essentially, assets and liabilities can be valued using general cash-flow model. Following Day's notation, the prospective model may be defined as:

$$\begin{aligned} V_a^0 &= CF_a^1(1 + i_a^1)^{-1} + CF_a^2(1 + i_a^2)^{-2} + \dots \\ &\quad + CF_a^t(1 + i_a^t)^{-t} + \dots \\ &= \sum_{j=1}^{\infty} CF_a^j(1 + i_a^j)^{-j} \end{aligned}$$

where:

V_a^t is the market value of asset a at time t .

CF_a^t is the cash flow generated by asset a at time t .

i_a^t is the discount rate applied to the cash flows generated by asset a as at time t .

Here this simplified model will consider only cash flows and discount rates, excluding mortality and other risk factors.

One of Day's claims is that actuaries enact manipulations on items such as cash flows that distort the true value of the data. "Actuaries tend to add together or otherwise algebraically manipulate expected values of stochastic variables such as cash flows.

When stochastic variables have different distributions (in magnitude or shape), then these are simply not additive" (Day 93). Day uses the zero-valued Bader swap to illustrate his point. A Bader swap is a debt-for-equity swap whereby an entity pays the return on a portfolio in exchange for receiving the return on an equity portfolio of equal size – i.e. swapping the current assets for an equal-valued portfolio (Bader 14).

Day's example consists of \$1 worth of short 15-year zero-coupon bonds yielding 6% and \$1 worth of long equities with dividends reinvested at an expected return of 8%

per year (with an expected standard deviation of 16% per year). Essentially, in a financial economic framework the cash-flow equation of this system is given by

$$\begin{aligned}
 V_{\text{baderswap}}^0 &= V_{\text{equities}}^0 - V_{\text{bonds}}^0 \\
 &= \sum_{j=0}^{\infty} CF_{\text{equities}}^j (1 + i_{\text{equities}})^{-j} \\
 &\quad - \sum_{j=0}^n CF_{\text{bonds}}^j (1 + i_{\text{bonds}})^{-j} \\
 &= 0.
 \end{aligned}$$

In contrast, the actuarial approach results in

$$\begin{aligned}
 V_{\text{equities}}^0 &= 1 = CF_{\text{equities}}^n (1 + i_{\text{equities}})^{-n} \\
 &\Rightarrow CF_{\text{equities}}^n = (1 + i_{\text{equities}})^n \\
 V_{\text{bonds}}^0 &= 1 = CF_{\text{bonds}}^n (1 + i_{\text{bonds}})^{-n} \\
 &\Rightarrow CF_{\text{bonds}}^n = (1 + i_{\text{bonds}})^n \\
 CF_{\text{baderswap}}^n &= CF_{\text{equities}}^n - CF_{\text{bonds}}^n \\
 &= (1 + i_{\text{equities}})^n - (1 + i_{\text{bonds}})^n.
 \end{aligned}$$

And thus

$$\begin{aligned}
 AV_{\text{baderswap}}^0 &= CF_{\text{baderswap}} * (1 + i_{\text{baderswap}})^{-15} \\
 &= [(1 + i_{\text{equities}})^n - (1 + i_{\text{bonds}})^n] / \\
 &\quad (1 + i_{\text{equities}})^n \\
 &= 1 - [(1 + i_{\text{bonds}})^n / (1 + i_{\text{equities}})^n].
 \end{aligned}$$

Thus, the actuarial present value is greater than zero (\$.024 in this example). Different modifications of this example will provide different results; however, each actuarial value results in a positive rather value rather than a zero-value.

The dilemma here is that “the Bader swap has an economic, market, real, theoretical, and practical value of zero” (Day 93). Whether cash flows are treated as stochastic or constant and however the discount rates are assumed, the significantly positive value achieved through the actuarial approach cannot be treated as the wholly correct answer.

While the differences in computational techniques may be lost on the average reader, the implications are a relevant concern to retirees. While computation is outside a retiree’s control, a retiree will still be affected by the decisions made in computation. For instance, an employer with a defined-contribution pension system using actuarial present value techniques will show a larger present-value than that of the financial economic method. Both are estimates of present value of a swap, but the actuarial present value is greater and may lead to larger benefits for the retirees in the plan throughout retirement. Therefore, this consideration and others – while high-level in nature – are relevant to retirees.

Conclusion

Retirement investment is an art. As with other financial ventures, there is no universal “ideal” investment strategy to be had. Rather, it is the duty of individuals, financial advisors, and trained professionals to take educated actions based on their knowledge of retirement vehicles and the financial markets. A greater and more thorough understanding of risks specifically related to retirement as well as available computational methods and tools will hopefully allow for more fruitful retirement strategies, allowing individuals to enjoy retirement more and worry about it less.

Appendix A: Centenarians

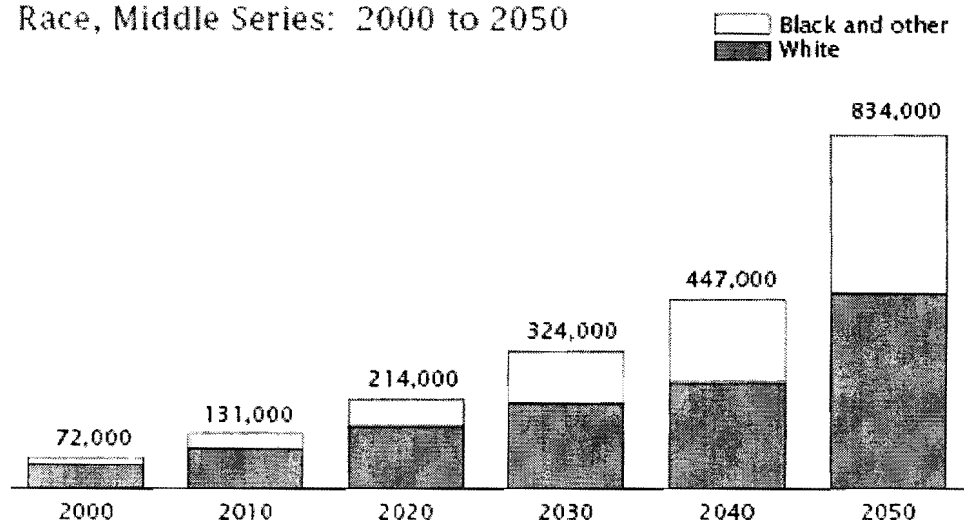
The following information is taken from a 1990 US Census Bureau survey analysis (Krach). The table below is an old-age cohort analysis survey. This table documents then-living persons over 100 years of age.

Cohort Analysis for 1990 Centenarians Aged 100 to 104: Born 1885 to 1890

Year	Age Range	Population (thousands)			Ratio of Males to females
		Total	Male	Female	
1890	under 5	7,635	3,885	3,750	103.6
1900	10 to 14	8,086	4,086	4,000	102.2
1910	20 to 24	9,117	4,613	4,504	102.4
1920	30 to 34	8,095	4,133	3,962	104.3
1930	40 to 44	8,052	4,166	3,886	107.2
1940	50 to 54	7,281	3,762	3,519	106.9
1950	60 to 64	6,103	3,058	3,045	100.4
1960	70 to 74	4,773	2,197	2,577	85.3
1970	80 to 84	2,312	883	1,429	61.8
1980	90 to 94	557	156	401	38.9
1990	100 to 104	31	6	25	24.0

The following graph estimates future US centenarian populations.

Number of Projected Centenarians by Race, Middle Series: 2000 to 2050

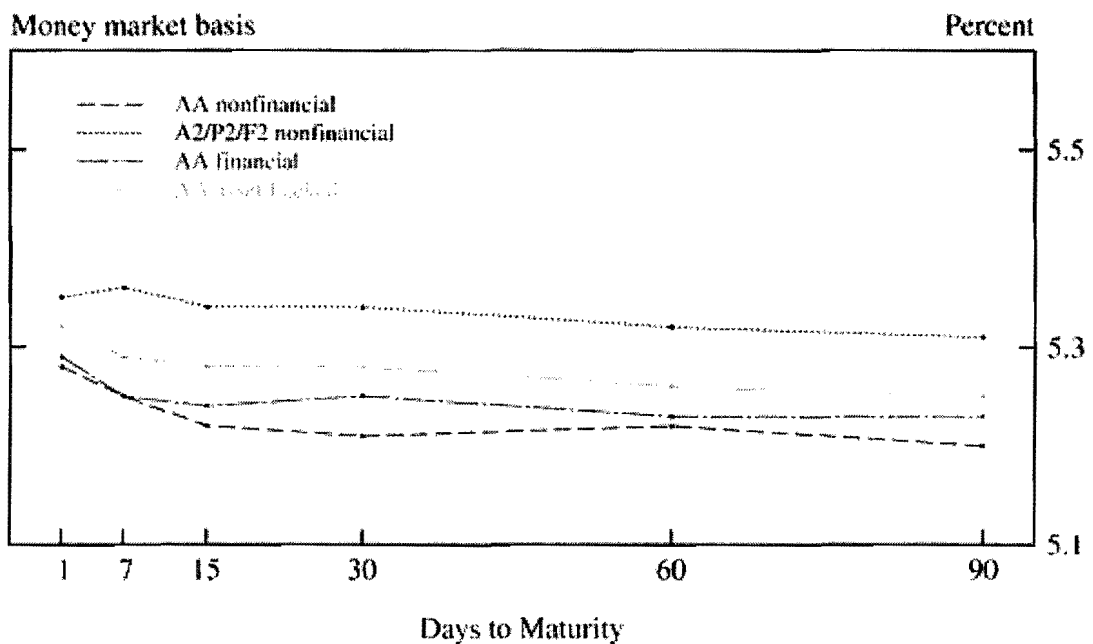


Appendix B: The Yield Curve

The following data and yield curve were retrieved from the US Federal Reserve web site (Commercial).

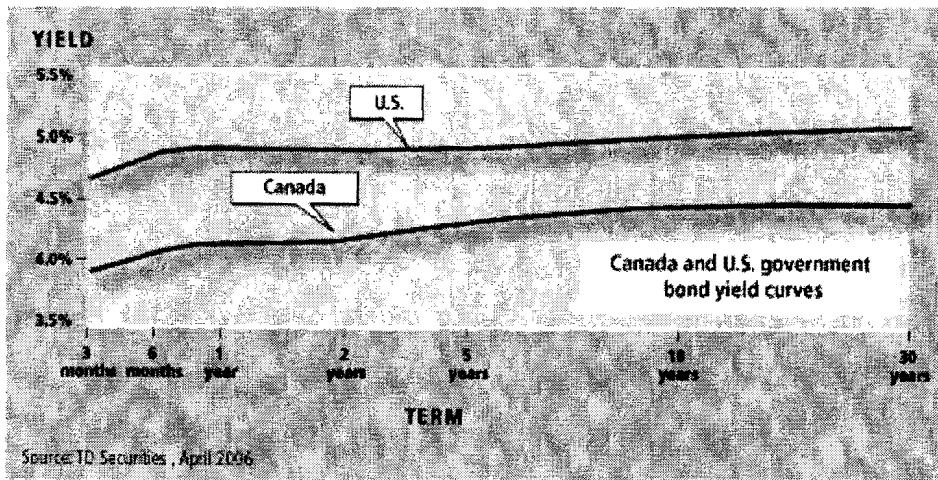
Commercial Paper Rates and Outstanding: Data as of April 17, 2007

Term	AA nonfinancial	A2/P2/F2 nonfinancial	AA financial	AA asset backed
1-day	5.28	5.35	5.29	5.32
7-day	5.25	5.36	5.25	5.29
15-day	5.22	5.34	5.24	5.28
30-day	5.21	5.34	5.25	5.28
60-day	5.22	5.32	5.23	5.26
90-day	5.20	5.31	5.23	5.25



The yield curve below highlights the flatness of actual yield curves from April 2006. The graph was retrieved from the TD Waterhouse Investment website (Wolanski).

Watching the Curve



Appendix C: Projected Medical Costs

The following table shows the decrease in projection results of a medical trend study (Segal 3). While most projections decreased in the 2006 study, the growth rates are still high and pose problems in funding future costs.

Projected Medical, Prescription Drug & Dental Trends: 2005 & 2006

Medical (Actives & Retirees < Age 65)	2005 Projected		2006 Projected	
	(without Rx)	(with Rx)*	(without Rx)	(with Rx)*
Indemnity Plans	14.4%	14.5%	14.4%	14.3%
Preferred Provider Organizations (PPOs)	12.6%	13.1%	12.4%	12.7%
Point-of-Service (POS) Plans	12.5%	13.0%	11.8%	12.2%
Health Maintenance Organizations (HMOs)	11.8%	12.4%	11.6%	12.0%
High-Deductible PPOs**	13.1%	13.5%	12.6%	12.8%

Medical (Retirees Age 65+)				
Medicare Supplemental Indemnity Plans	10.1%	11.1%	9.5%	11.2%
Medicare Advantage (MA) Plans***	9.1%	10.3%	8.8%	10.7%

Prescription Drug (Rx) Carve-Out (Actives & Retirees < Age 65)		
Retail	15.2%	13.8%
Mail Order	15.5%	14.5%

Rx Carve-Out (Retirees Age 65+)		
Retail	15.6%	14.2%
Mail Order	15.4%	14.3%

Dental		
Indemnity Plans	7.1%	7.0%
Dental Provider Organizations (DPOs)	6.7%	6.3%
Dental Maintenance Organizations (DMOs)	4.8%	5.2%

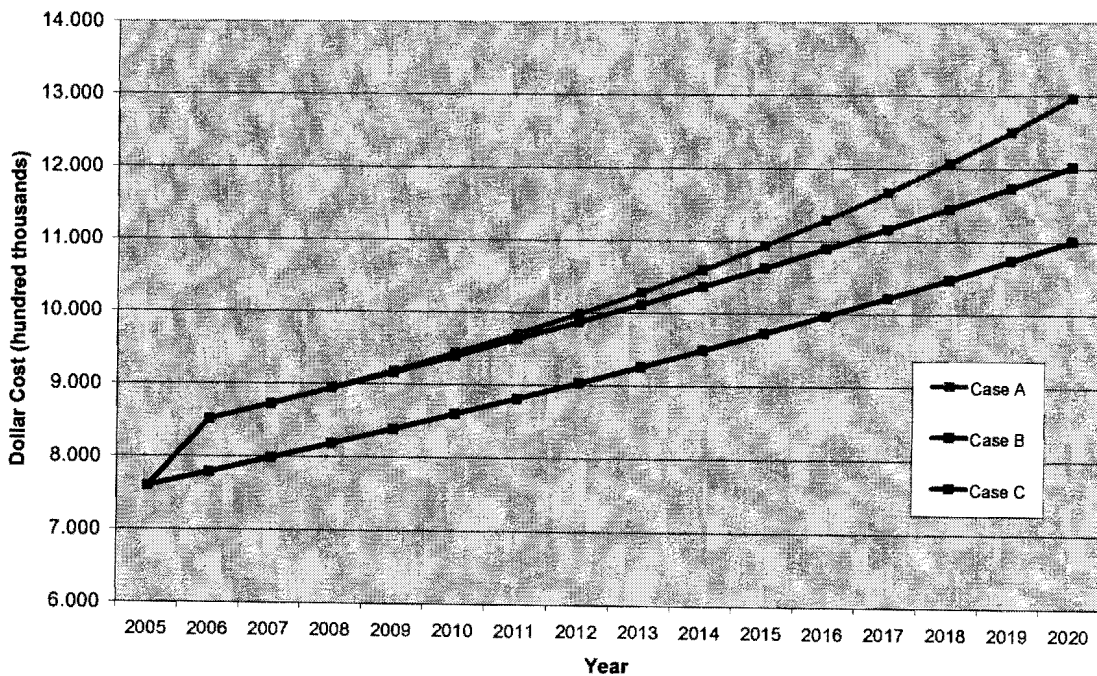
* Trend projections were derived by proportionally blending medical plan trends and freestanding prescription drug trends.

** High-deductible PPOs are defined as those with a minimum deductible of \$1,000.

*** The Medicare Prescription Drug, Improvement, and Modernization Act of 2003 changed the official name of Medicare managed plans from "Medicare+Choice" plans to "Medicare Advantage" (MA) plans.

The following table illustrates the example on page 13 of this paper. Case A represents an annual inflation of 2.5 percent; Case B represents a first-year increase of 12 percent followed by an annual inflation of 2.5 percent; Case C represents a first-year increase of 12 percent followed by an initial annual inflation of 2.5 percent which increases by 3 percent of prior year inflation thereafter.

Example Health Care Increase (Inflation Only)



Appendix D: Investment Strategies

The following numerical example was taken directly from Moore and Young (2006), using the following base scenario:

- Use the Gompertz hazard rate $\lambda_x(t) = \exp \frac{x+t-\hat{m}}{b} / b$. We choose $\hat{m} = 90$ and $b = 9$; these values approximate the Individual Annuity Mortality 2000 (basic) Table with projection scale G. Note that the hazard rate increases exponentially with age.
- $x = 50$; the investor is 50 years old. Under the mortality assumption described above, her expected future lifetime is 35.32 years.
- $r = 0.02$; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$; the drift on the risky asset is 6% over inflation.
- $\sigma = 0.20$; the volatility of the risky asset is 20%.
- $c = 1$; the individual consumes one unit of real wealth per year.
- $w_l = 0$; the individual considers herself ruined when her wealth reaches 0.
- $A = 0$; without loss of generality, we assume that annuity income is zero.
- It follows that the annual shortfall is $c - A = 1$. The individual is safe from ruin when wealth reaches $w_a = \frac{c - A}{r} = 50$.

Figure 1

Ruin Probabilities and Optimal Investment Strategies as Attained Age x Is Varied

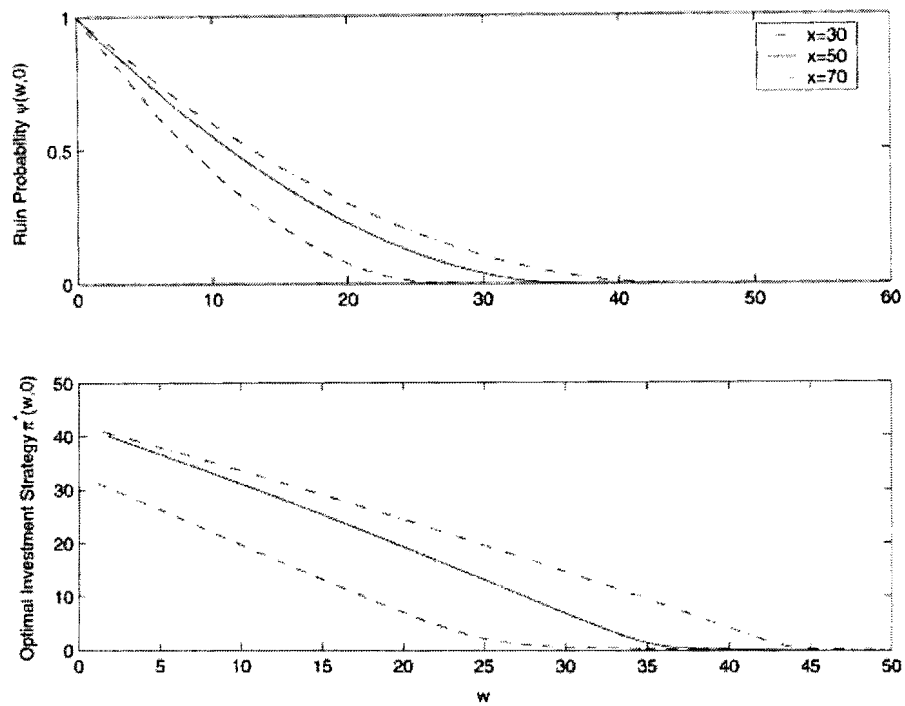
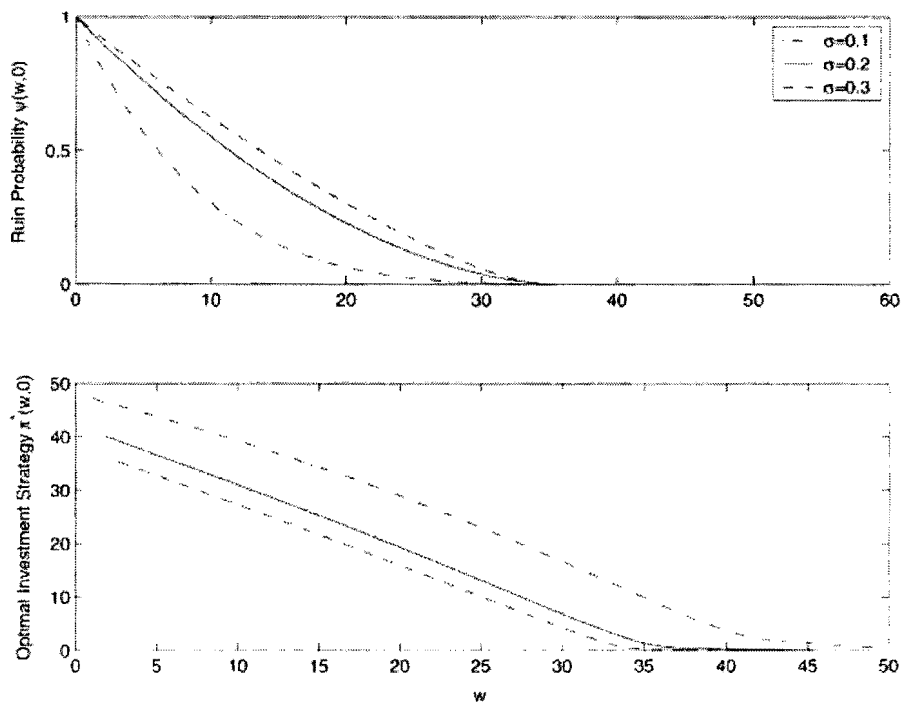


Figure 2

Ruin Probabilities and Optimal Investment Strategies as Volatility σ of the Risky Asset Is Varied



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